QUANTUM PRINCIPAL BUNDLES & TANNAKA-KREIN DUALITY THEORY

MIĆO ĐURĐEVIĆ

Abstract. The structure of quantum principal bundles is studied, from the viewpoint of Tannaka-Krein duality theory. It is shown that if the structure quantum group is compact, principal G-bundles over a quantum space M are in a natural correspondence with certain contravariant functors defined on the category of finite-dimensional unitary representations of G, with the values in the category of finite projective bimodules over a *-algebra representing the base space.

1. Introduction

The aim of this paper is to study the internal structure of quantum principal bundles possessing compact structure groups, from the point of view of Tannaka-Krein duality theory. This analysis can be understood as a non-commutative generalization of the classical incorporation of the structure of principal bundles in the language of corresponding associated vector bundles.

The structure of a compact matrix quantum group G is completely encoded [2, 3] in the category $R(G)$ of finite-dimensional unitary representations of G (corresponding to G, in the framework of Tannaka-Krein duality). The main result of this paper is that there exists a natural correspondence between quantum principal G-bundles P over a quantum space M, and appropriate contravariant functors τ_P on $R(G)$, with values in the category Φ_M of finite projective bimodules over the *-algebra $\mathcal{V}$ representing the base M. Explicitly, these bimodules consist of intertwiners between representations from $R(G)$ and the right action of G on P. They are interpretable as quantum counterparts of associated vector bundles.

In non-commutative differential geometry [1], vector bundles can be naturally viewed as one-sided finite-projective modules. Such a concept of a vector bundle is especially natural in considerations involving cyclic cohomology. In particular, the standard algebraic K-theory naturally enters the game. However, associated vector bundles introduced in this paper are much more rigid objects, in particular they always appear as *bimodules*.

The paper is organized as follows. The next section describes a construction of the functor τ_P, starting from a quantum principal bundle P. In Section 3 the inverse construction is presented, which reconstructs the bundle starting from a contravariant functor $\tau\colon R(G) \to \Phi_M$ possessing necessary additional properties. In Section 4 some examples are considered.

It is worth noticing that even if M and G are classical the bundle P may be a

quantum object. In particular, an interesting purely quantum phenomena is the nontriviality of the classification for bundles over a 1-point set.

The important point is that functors τ do not see the "concrete" aspect of the category $R(G)$, given by specifying carrier unitary spaces of representations. This means that the classification problem for groups having "abstractly" the same categories of representations will be the same.

2. Structural Analysis of Quantum Principal Bundles

Let G be a compact matrix quantum group [2], represented by a Hopf *-algebra $\mathcal{A}$, consisting of "polynomial functions" on G. Let $\phi\colon \mathcal{A}\rightarrow\mathcal{A}\otimes\mathcal{A}$, $\epsilon\colon\mathcal{A}\rightarrow\mathbb{C}$ and $\kappa\colon\mathcal{A}\rightarrow\mathcal{A}$ be the coproduct, counit and the antipode map.

In the framework of Tannaka-Krein duality theory [3], the group G is represented by a complete concrete monoidal W-category $R(G)$. The objects of $R(G)$ are unitary representations of G in finite-dimensional unitary spaces. Morphisms in $R(G)$ are intertwiners between representations.

For each $u\in R(G)$ we shall denote by H_u the corresponding carrier space (so that u is understandable as a map $u\colon H_u\rightarrow H_u\otimes\mathcal{A}$). We shall denote by $\oplus$, $\times$ the sum and the product in $R(G)$. In particular,

$$H_{u\oplus v}=H_u\oplus H_v\qquad H_{u\times v}=H_u\otimes H_v.$$

We shall denote by $\varnothing$ the trivial representation of G, acting in $H_\varnothing=\mathbb{C}$. In what follows we may assume that spaces H_u belong to a given model set of unitary spaces, so that $R(G)$ is small.

Let us consider a quantum space M, represented by a unital *-algebra $\mathcal{V}$. Let $P=(\mathcal{B},i,F)$ be a quantum principal G-bundle [5] over M. Here $\mathcal{B}$ is a *-algebra consisting of appropriate "functions" on P, while $F\colon\mathcal{B}\rightarrow\mathcal{B}\otimes\mathcal{A}$ and $i\colon\mathcal{V}\rightarrow\mathcal{B}$ are unital *-homomorphisms playing the role of the "dualized" right action of G on P, and the dualized projection of P on M respectively. We have

$$(F\otimes\mathrm{id})F=(\mathrm{id}\otimes\phi)F\qquad(\mathrm{id}\otimes\epsilon)F=\mathrm{id}\qquad i(\mathcal{V})=\Bigl\{\,b\in\mathcal{B};F(b)=b\otimes 1\,\Bigr\}.$$

Let us consider the intertwiner $\mathcal{V}$-bimodules

$$\mathcal{E}_u=\mathrm{Mor}(u,F).$$

The elements of these bimodules are interpretable as smooth sections of corresponding associated vector bundles.

From the analysis of [5]–Appendix A it follows that $\mathcal{E}_u$ are (both side) finite and projective. Each morphism $\lambda\in\mathrm{Mor}(u,v)$ induces a bimodule homomorphism $\varrho_{uv}(\lambda)\colon\mathcal{E}_v\rightarrow\mathcal{E}_u$, via the composition of intertwiners. In other words, we have a natural system of linear maps $\varrho_{uv}\colon\mathrm{Mor}(u,v)\rightarrow\mathrm{Hom}(\mathcal{E}_v,\mathcal{E}_u)$. In what follows we shall also use a simpler alternative notation and write

$$\lambda_\star=\varrho_{uv}(\lambda),$$

when the domain/codomain is clear from the context.

We have

$$\lambda_\star\rho_\star=[\rho\lambda]_\star\tag{2.1}$$

for $\lambda\in\mathrm{Mor}(u,v)$ and $\rho\in\mathrm{Mor}(v,w)$.

For each $u \in R(G)$ let $\bar{u} = u^c$ be the corresponding conjugate representation, identified with the contragradient representation. By definition,

$$H_{\bar{u}} = H_u^* \qquad u^c = (\mathrm{id} \otimes \kappa)(u)^\top.$$

Let $C_u \colon H_u \to H_u$ be the canonical intertwiner between u and u^{cc}. This is a strictly positive map, normalized such that $\mathrm{tr}(C_u) = \mathrm{tr}(C_u^{-1})$. Explicitly, C_u can be described as follows. Let $(f_z)_{z\in\mathbb{C}}$ be a canonical holomorphic family of linear multiplicative functionals $f_z \colon \mathcal{A} \to \mathbb{C}$, describing modular properties of the Haar measure [2]. Then we have

$$C_u = (\mathrm{id} \otimes f_1)u.$$

In particular, it follows that

$$C_u \otimes C_v = C_{u\times v} \qquad \varphi C_u = C_v \varphi$$

for each $\varphi \in \mathrm{Mor}(u, v)$ and $u, v \in R(G)$. The natural scalar product in H_u^* is given by

$$(f, g) = (j_u^{-1}(g), C_u j_u^{-1}(f))$$

where $j_u \colon H_u \to H_u^*$ is the canonical antilinear map (induced by the scalar product in H_u). The representation $\bar{u}$ is unitary, with respect to the above scalar product. Furthermore, we have

$$C_u = j_{\bar{u}} j_u$$

for each $u \in R(G)$.

In what follows, a natural identification $(W \otimes V)^* = V^* \otimes W^*$ will be assumed. Then

$$j_v(y) \otimes j_u(x) = j_{u\times v}(x \otimes y) \qquad (u \times v)^c = v^c \times u^c$$

for each $u, v \in R(G)$.

For each $f \in \mathrm{Mor}(u, v)$ we shall denote by $f^c \in \mathrm{Mor}(\bar{u}, \bar{v})$ a morphism specified by

$$f^c j_u = j_v f. \tag{2.2}$$

To every representation $u \in R(G)$ we can associate two elementary intertwiners $I_u \in \mathrm{Mor}(\varnothing, u\bar{u})$ and $\gamma^u \in \mathrm{Mor}(\bar{u}u, \varnothing)$, given by the identity operator and the contraction map of H_u respectively. Because of the symmetry between u and $\bar{u}$ we have also intertwiners $I^u \in \mathrm{Mor}(\varnothing, \bar{u}u)$ and $\gamma_u \in \mathrm{Mor}(u\bar{u}, \varnothing)$. In explicit form,

$$\gamma^u(f \otimes x) = f(x) \qquad \gamma_u(x \otimes f) = fC_u(x)$$
$$I_u(1) = \sum_i e_i \otimes e_i^* \qquad I^u(1) = \sum\nolimits_{ij} [C_u^{-1}]_{ji} e_i^* \otimes e_j,$$

where $\{e_i\}$ is an arbitrary orthonormal basis in H_u and $\{e_i^*\}$ is the corresponding biorthogonal basis.

We shall use the symbol $\otimes_M$ for the tensor product over $\mathcal{V}$, in order to stress the geometrical background for our considerations.

Proposition 2.1. *The following natural bimodule isomorphism holds*

$$\mathcal{E}_{u\times v} \leftrightarrow \mathcal{E}_u \otimes_M \mathcal{E}_v. \tag{2.3}$$

The above isomorphism is induced by the product map of $\mathcal{B}$.

Proof. Let us fix a representation u. As follows from [5]-Appendix A, there exist intertwiners $\mu_k, \nu_k \in \mathcal{E}_{\bar{u}}, \mathcal{E}_u$ such that

$$\sum_k \mu_k(f)\nu_k(x) = f(x)1,$$

for each $f \in H_{\bar{u}}$ and $x \in H_u$.

Let $\vartheta_{vu} \colon \mathcal{E}_{v\times u} \to \mathcal{E}_v \otimes_M \mathcal{E}_u$ be a map given by

$$\vartheta_{vu}(\xi) = \sum_k \xi_k \otimes \nu_k,$$

where $\xi_k(f) = \sum_j \xi(f \otimes e_j)\mu_k(e_j^*)$.

We prove that the inverse of ϑ_{vu} is given by $\vartheta_{vu}^{-1}(\varphi \otimes \psi)(y \otimes x) = \varphi(y)\psi(x)$. A direct calculation gives

$$\vartheta_{vu}^{-1}\vartheta_{vu}(\xi)(y \otimes x) = \sum_{kj} \xi(y \otimes e_j)\mu_k(e_j^*)\nu_k(x) = \xi(y \otimes x),$$

and similarly

$$\vartheta_{vu}\vartheta_{vu}^{-1}(\varphi \otimes \psi) = \sum_{kj} \varphi\bigl\{\psi(e_j)\mu_k(e_j^*)\bigr\} \otimes \nu_k = \sum_{kj} \varphi \otimes \psi(e_j)\mu_k(e_j^*)\nu_k = \varphi \otimes \psi,$$

and hence ϑ_{vu} is bijective. $\square$

In terms of the above identification, the elementary intertwiners I_u and γ^u induce bimodule homomorphisms

$$\gamma_\star^u \colon \mathcal{V} \to \mathcal{E}_{\bar{u}} \otimes_M \mathcal{E}_u \qquad \langle\rangle_u^+ = [I_u]_\star \colon \mathcal{E}_u \otimes_M \mathcal{E}_{\bar{u}} \to \mathcal{V}$$

respectively. Explicitly, it follows that

$$\gamma_\star^u(1) = \sum_k \mu_k \otimes \nu_k \qquad \langle \varphi \otimes \psi\rangle_u^+ = \sum_i \varphi(e_i)\psi(e_i^*). \tag{2.4}$$

Interchanging the roles of u and $\bar{u}$ we obtain a pairing $\langle\rangle_u^- \colon \mathcal{E}_{\bar{u}} \otimes_M \mathcal{E}_u \to \mathcal{V}$ and the injection $\gamma_u^\star \colon \mathcal{V} \to \mathcal{E}_u \otimes_M \mathcal{E}_{\bar{u}}$.

Maps j_u naturally induce antilinear antiisomorphisms $\sharp_u \colon \mathcal{E}_u \to \mathcal{E}_{\bar{u}}$, via the following diagram

$$\begin{tikzcd} H_u \arrow[r, "\varphi"] \arrow[d, "j_u"'] & \mathcal{B} \arrow[d, "*"] \\ H_u^* \arrow[r, "\sharp_u\varphi"'] & \mathcal{B} \end{tikzcd} \tag{2.5}$$

With the help of $\sharp_u$, the bimodule $\mathcal{E}_{\bar{u}}$ is naturally identificable with the *conjugate* of $\mathcal{E}_u$.

Lemma 2.2. *The following identities hold*

$$\sharp_{\bar{u}}\sharp_u = [C_u^{-1}]_\star$$
$$f^c_\star = \sharp_u f_\star \sharp_v^{-1}$$
$$\sharp_v(\psi)\otimes\sharp_u(\varphi) = \sharp_{u\times v}(\varphi\otimes\psi),$$

where $f \in \mathrm{Mor}(u,v)$ *and* $\varphi,\psi \in \mathcal{E}_{u,v}$. $\square$

Let us observe that the first two equalities allow us to extend the introduced representation of morphisms, to all *antilinear morphisms*. By definition, an antilinear map $f\colon H_u \to H_v$ is a *morphism*, iff the diagram

$$\begin{array}{ccc} H_u & \xrightarrow{\;u\;} & H_u\otimes\mathcal{A} \\ f\big\downarrow & & \big\downarrow f\otimes * \\ H_v & \xrightarrow[\;v\;]{} & H_v\otimes\mathcal{A} \end{array} \tag{2.6}$$

is commutative. Let us denote by $\bar{\mathrm{M}}(u,v)$ the corresponding spaces of antilinear morphisms. From this moment, we shall include all such maps in the system of morphisms of $R(G)$. Consequently, let us assume that

$$\mathrm{M}(u,v) = \mathrm{Mor}(u,v)\oplus\bar{\mathrm{M}}(u,v)$$

represent arrows of $R(G)$. These spaces will be endowed with their natural $\mathbb{Z}_2$-grading.

Let Φ_M be the category of finite projective $\mathcal{V}$-bimodules. For each $\mathcal{J},\mathcal{K}\in\Phi_M$, let $\bar{\mathrm{H}}(\mathcal{J},\mathcal{K})$ be the space of the corresponding antilinear bimodule antihomomorphisms, and let us consider the spaces

$$\mathrm{H}(\mathcal{J},\mathcal{K}) = \mathrm{Hom}(\mathcal{J},\mathcal{K})\oplus\bar{\mathrm{H}}(\mathcal{J},\mathcal{K}),$$

endowed with the natural $\mathbb{Z}_2$-grading. We shall assume that the arrows of Φ_M are given by the elements of $\mathrm{H}(\mathcal{J},\mathcal{K})$.

If $f\in\bar{\mathrm{M}}(u,v)$ then the formula

$$f_\star = [j_v f]_\star \sharp_v \tag{2.7}$$

defines a grade-preserving linear extension $\varrho_{uv}\colon \mathrm{M}(u,v)\to\mathrm{H}(\mathcal{E}_u,\mathcal{E}_v)$ of the previously introduced representation. The formula (2.1) remains valid for arbitrary $f,g\in\mathrm{M}(u,v)$.

Let $\mathcal{T}\subseteq R(G)$ be a complete set of mutually inequivalent irreducible unitary representations of G. The algebra $\mathcal{B}$ can be decomposed into a direct sum of multiple irreducible $\mathcal{V}$-bimodules

$$\mathcal{B} = \sum_{\alpha\in\mathcal{T}}^{\oplus}\mathcal{B}^\alpha,$$

relative to the right action F. Each $\mathcal{B}^\alpha$ can be further decomposed as

$$\mathcal{B}^\alpha = \mathrm{Mor}(\alpha,F)\otimes H_\alpha \qquad \varphi(x)\leftrightarrow\varphi\otimes x.$$

In this sense $\mathcal{E}_\alpha = \mathrm{Mor}(\alpha,F)$ are elementary building blocks of the bundle P.

Motivated by the above considerations, we are going to formulate a notion of a representation of a concrete monoidal W-category.

Let $R \subseteq R(G)$ be a generating monoidal subcategory, closed (up to the equivalence) under the conjugation functor and containing all the conjugation maps.

(*rep1*) Let us assume that we have a pair

$$\varrho = \Bigl(\Bigl\{\mathcal{E}_u; u \in R\Bigr\}\Bigl\{\varrho_{uv}; u, v \in R\Bigr\}\Bigr)$$

where $\mathcal{E}_u$ are finite projective $\mathcal{V}$-bimodules, $\varrho_{uv}\colon \mathrm{M}(u,v) \to \mathrm{H}(\mathcal{E}_v, \mathcal{E}_u)$ are grade-preserving linear maps, realizing a contravariant functor $\varrho\colon R \to \Phi_M$. We shall also use the simplified notation $\varrho_{uv}(f) = f_\star$.

Let us consider the category $R \times R$ defined in the following way. The objects are the corresponding ordered pairs, while the morphisms between (u,p) and (v,q) are pairs (φ,ψ) satisfying $\varphi \in \mathrm{Mor}(u,v)$ and $\psi \in \mathrm{Mor}(p,q)$ for the degree 0, and satisfying $\varphi \in \bar{\mathrm{M}}(u,q)$ and $\psi \in \bar{\mathrm{M}}(p,v)$ for the degree 1. In a similar way, let us introduce the morphisms in $\Phi_M \times \Phi_M$. Let us denote by $[\varphi,\psi]$ the twisted tensor product of the corresponding antilinear transformations. The operations $\times$ and $\otimes_M$ are undersandable as grade-preserving covariant functors

$$\times\colon R \times R \to R \qquad \otimes_M\colon \Phi_M \times \Phi_M \to \Phi_M,$$

transforming (φ,ψ) into $\varphi \otimes \psi$ or $[\varphi,\psi]$, depending on the parity.

(*rep2*) Furthermore, let us assume that there exists a natural transformation ϑ between functors $\varrho \circ \{\times\}$ and $\otimes_M \circ (\varrho \times \varrho)$, given by the system of bimodule isomorphisms $\vartheta_{uv}\colon \mathcal{E}_{u\times v} \to \mathcal{E}_u \otimes_M \mathcal{E}_v$.

(*rep3*) Finally, let us assume that ϑ is associative, in the sense that

$$(\vartheta_{uv} \otimes \mathrm{id})\vartheta_{u\times v,w} = (\mathrm{id} \otimes \vartheta_{vw})\vartheta_{u,v\times w}, \tag{2.8}$$

for each $u, v, w \in R$.

Definition 2.1. Every pair $\tau = (\varrho, \vartheta)$ satisfying the above listed properties is called a *representation* of R in Φ_M.

The analysis of this section may be now summarized as follows

Proposition 2.3. *Let $P = (\mathcal{B}, i, F)$ be an arbitrary quantum principal G-bundle over M. Then the corresponding intertwiner bimodules $\mathcal{E}_u$, together with the associated system of maps ϱ_{uv} and the product identifications form a representation τ_P of $R(G)$ in Φ_M.* $\square$

3. Bundle Reconstruction

The aim of this section is to prove that the construction of the previous section works in both directions, so that there exists a natural bijection

$$\Bigl\{\text{Quantum } G\text{-bundles } P \text{ over } M\Bigr\} \leftrightarrow \Bigl\{\text{Representations } \tau \text{ of } R(G) \text{ in } \Phi_M\Bigr\}.$$

Let R be a generating subcategory of $R(G)$, closed (up to the equivalence) under taking conjugate objects.

Proposition 3.1. *Every representation of R in Φ_M can be extended to a representation of $R(G)$. The extension is unique, up to a natural transformation.*

Proof. Let τ be a representation of R in Φ_M. Let us first assume that the extension exists, and prove its uniqueness. Each object $u \in R(G)$ can be realized as a direct summand in some $r = \sum_k^{\oplus} r_k$, where $r_k \in R$, with the help of embedding and projection morphisms, $\iota_u \in \mathrm{Mor}(u, r)$ and $\pi_u \in \mathrm{Mor}(r, u)$. Then the object $\mathcal{E}_u$ is realizable as a submodule in $\mathcal{E}_r = \sum_k^{\oplus} \mathcal{E}_{r_k}$, with the help of the embedding $[\pi_u]_\star$ and the projection $[\iota_u]_\star$.

Let us consider a representation v, realized in a similar way in $s = \sum_k^{\oplus} s_k$. For each $f \in \mathrm{Mor}(u, v)$ the bimodule homomorphism $f_\star$ is uniquely fixed, because we can write $f \leftrightarrow \sum_{kl} i_l \rho_{kl} p_k$, where $i_l \in \mathrm{Mor}(s_l, s)$ and $p_k \in \mathrm{Mor}(r, r_k)$ are canonical embedings and projections, and $\rho_{kl} \in \mathrm{Mor}(r_k, s_l)$. This means that all maps $f_\star$ are expressible in terms of R. The same holds for antilinear morphisms. Hence, the extension is unique.

Conversely, starting from the above observations it is possible to *define* the extension of τ. Objects and morphisms of $R(G)$ are expressible purely in terms of R, and such expressions consistently define the extension of τ. In particular, bimodules $\mathcal{E}_u$ can be invariantly described by "gluing" the corresponding images in various possible $\mathcal{E}_r$. We omit technical details. $\square$

Let τ be an arbitrary representation of $R(G)$ in Φ_M. We are going to re-construct the bundle P which satisfies $\tau = \tau_P$.

As first, let us define a $\mathcal{V}$-bimodule $\mathcal{B}$ as a direct sum

$$\mathcal{B} = \sum_{\alpha \in \mathcal{T}}^{\oplus} \mathcal{E}_\alpha \otimes H_\alpha. \tag{3.1}$$

The group G naturally acts on $\mathcal{B}$ on the right, via the sum F of actions $\mathrm{id} \otimes \alpha$. Let $i \colon \mathcal{V} \to \mathcal{B}$ be the canonical inclusion map. By definition, $b \in \mathrm{im}(i)$ iff $F(b) = b \otimes 1$ for each $b \in \mathcal{B}$.

Let us further observe that the following natural isomorphism holds

$$\mathcal{E}_u \leftrightarrow \mathrm{Mor}(u, F), \tag{3.2}$$

for each $u \in R(G)$. Indeed, if $u = \alpha \in \mathcal{T}$ then the above correspondence is simply $\varphi \colon x \mapsto \varphi \otimes x$.

Let us consider an arbitrary representation $u \in R(G)$. The following intrinsic bimodule decomposition holds

$$\mathcal{E}_u = \sum_{\alpha \in \mathcal{T}}^{\oplus} \mathrm{Mor}(u, \alpha) \otimes \mathcal{E}_\alpha, \tag{3.3}$$

where actually the sum is finite. On the other hand

$$\mathrm{Mor}(u, F) = \sum_{\alpha \in \mathcal{T}}^{\oplus} \mathrm{Mor}(u, \alpha) \otimes \mathrm{Mor}(\alpha, F), \tag{3.4}$$

where the identification is induced by the composition of morphisms. Combining the above decompositions, we conclude that (3.2) holds in the full generality. In terms of this identification, the following correspondence holds

$$f_\star(\varphi) \leftrightarrow \varphi f, \tag{3.5}$$

for each $u, v \in R(G)$, $\varphi \in \mathcal{E}_v$ and $f \in \mathrm{Mor}(u, v)$. In what follows we shall assume all the above identifications.

For $\varphi \in \mathcal{E}_u$ and $\psi \in \mathcal{E}_v$ we shall denote by $\varphi\psi \in \mathcal{E}_{u\times v}$ the element given by

$$\varphi\psi = \vartheta_{uv}^{-1}(\varphi \otimes \psi). \tag{3.6}$$

The associativity of the natural transformation ϑ implies that the above product is associative, too.

Proposition 3.2. *The formula*

$$\varphi(x)\psi(y) = (\varphi\psi)(x \otimes y) \tag{3.7}$$

where $\varphi \in \mathcal{E}_u$ and $\psi \in \mathcal{E}_v$, consistently defines a structure of a unital associative algebra in $\mathcal{B}$ so that maps $\{\, i, F \,\}$ are unital homomorphisms.

Proof. The above formula, restricted to representations from $\mathcal{T}$, defines a bilinear product in $\mathcal{B}$, so that the embedding i is multiplicative. The fact that this product satisfies the same formula for arbitrary representations follows from the naturality of ϑ. Further, $i(1) = 1 \otimes 1$ is the unit element. In particular F is unital, too. The map F preserves the introduced product because

$$\begin{aligned} F\bigl(\varphi(e_i)\psi(e_j)\bigr) &= F\bigl((\varphi\psi)(e_i \otimes e_j)\bigr) = \sum_{kl} (\varphi\psi)(e_k \otimes e_l) \otimes u_{ki}v_{lj} \\ &= \sum_{kl} \varphi(e_k)\psi(e_l) \otimes u_{ki}v_{lj} = \sum_{kl} \bigl(\varphi(e_k) \otimes u_{ki}\bigr)\bigl(\psi(e_l) \otimes v_{lj}\bigr) = F\varphi(e_i)F\psi(e_j). \end{aligned}$$

As a consequence of the associativity of the product of intertwiners we obtain

$$\begin{aligned} \varphi(x)\bigl(\psi(y)\eta(z)\bigr) &= \varphi(x)\bigl((\psi\eta)(y \otimes z)\bigr) = \bigl(\varphi(\psi\eta)\bigr)(x \otimes y \otimes z) \\ &= \bigl((\varphi\psi)\eta\bigr)(x \otimes y \otimes z) = (\varphi\psi)(x \otimes y)\eta(z) = \bigl(\varphi(x)\psi(y)\bigr)\eta(z), \end{aligned}$$

which completes the proof. $\square$

Now we shall define the *-structure on the algebra $\mathcal{B}$. There exists the unique antilinear map $*\colon \mathcal{B} \to \mathcal{B}$ satisfying

$$[f_\star(\psi)(x)]^* = \psi[f(x)]$$

for each $f \in \bar{\mathrm{M}}(v, u)$ and $\psi \in \mathcal{E}_u$. The consistency of this definition directly follows from the functoriality of τ.

Proposition 3.3. *The introduced map is a *-structure on the algebra $\mathcal{B}$.*

Proof. Let $\sharp_u\colon \mathcal{E}_u \to \mathcal{E}_{\bar u}$ be antilinear bimodule homomorphisms given by

$$\sharp_u = [j_u^{-1}]_\star.$$

We have then

$$\psi(x)^{**} = \bigl[(\sharp_u\psi)(j_u x)\bigr]^* = (\sharp_{\bar u}\sharp_u\psi)(j_{\bar u}j_u x) = \bigl\{[C_u^{-1}]_\star\psi\bigr\}(C_u x) = \psi(x),$$

which shows that $*$ is involutive. Furthermore, if $\varphi \in \mathcal{E}_u$ and $\psi \in \mathcal{E}_v$ then

$$\begin{aligned}\bigl(\varphi(x)\psi(y)\bigr)^* &= \bigl[\sharp_{u\times v}(\varphi\psi)\bigr](j_v(y)\otimes j_u(x))\\ &= \bigl(\sharp_v\psi\bigr)\bigl(j_v(y)\bigr)\bigl(\sharp_u\varphi\bigr)\bigl(j_u(x)\bigr) = \psi(y)^*\varphi(x)^*,\end{aligned}$$

in accordance with the naturality of the transformation ϑ. Hence, $*$ is antimultiplicative. □

Summarizing the analysis of this section, we conclude that

Theorem 3.4. *The triplet $P = (\mathcal{B}, i, F)$ is a quantum principal G-bundle over M. Moreover, $\tau = \tau_P$.*

Proof. It remains to prove that G is acting freely on P. For a given $u \in R(G)$ let us consider elements $\mu_k \in \mathcal{E}_{\bar u}$ and $\nu_k \in \mathcal{E}_u$ such that

$$\gamma_\star^u(1) \leftrightarrow \sum\nolimits_k \mu_k \otimes \nu_k.$$

We have then

$$\sum_k \mu_k j_u(x)\nu_k(y) = (x,y)1,$$

for each $x, y \in H_u$. This implies that F is free. □

4. Examples

4.1. Bundles Over 1-point Sets

If M is a 1-point set then $\mathcal{V} = \mathbb{C}$ and objects of Φ_M are complex finite-dimensional vector spaces. Morphisms are linear maps, and binary operations are standard direct sums and tensor products of spaces. The problem of classifying G-bundles over such a trivial space M thus reduces to a linear-algebraic game. Interestingly, all non-trivial examples of bundles of this type will be completely quantum–without points at all.

4.2. Quantum Line Bundles

Let us assume that $G = U(1)$. Then $\mathcal{A}$ is generated by a single unitarity u satisfying $\phi(u) = u\otimes u$, identified with the fundamental representation. Irreducible representations are labeled by integers, $u^n \leftrightarrow n$. Category $R(G)$ is generated by objects $\bar u = u^{-1}$ and u. The only relations in $R(G)$ are identifications

$$\bar u \times u \leftrightarrow \varnothing \qquad u \times \bar u \leftrightarrow \varnothing,$$

together with the compatibility conditions

$$\begin{gathered}(\bar u \times u)\times \bar u \leftrightarrow \varnothing \times \bar u = \bar u \times \varnothing \leftrightarrow \bar u \times (u \times \bar u)\\ (u \times \bar u)\times u \leftrightarrow \varnothing \times u = u \times \varnothing \leftrightarrow u \times (\bar u \times u).\end{gathered}$$

Let τ be a representation of $R(G)$ in Φ_M. It follows that τ is completely specified by a $\mathcal{V}$-bimodule $\mathcal{E}$ and its conjugate $\bar{\mathcal{E}}$, together with hermitian bimodule isomorphisms $\bar{\mu}\colon \mathcal{E}\otimes_M\bar{\mathcal{E}}\to\mathcal{V}$ and $\mu\colon \bar{\mathcal{E}}\otimes_M\mathcal{E}\to\mathcal{V}$, which are mutually compatible such that

$$\mu\otimes\mathrm{id}=\mathrm{id}\otimes\bar{\mu}\qquad \bar{\mu}\otimes\mathrm{id}=\mathrm{id}\otimes\mu.$$

As a concrete illustration, let us assume that M is a classical compact smooth manifold, and $\mathcal{V}=S(M)$. Then, at the level of left modules, the elements of $\mathcal{E}$ can be naturally identified with smooth sections of a line bundle L over M. In terms of this identification the right $\mathcal{V}$-module structure is determined by a *-automorphism $\varepsilon\colon S(M)\to S(M)$, so that $\varphi f=\varepsilon(f)\varphi$, for each $\varphi\in\mathcal{E}$ and $f\in S(M)$. The presence of ε does not influence the bimodule structure of $\mathcal{E}\otimes_M\bar{\mathcal{E}}$ and the vector space structure of $\bar{\mathcal{E}}\otimes_M\mathcal{E}$. Hence it is possible to introduce classical natural maps $\mu_0\colon\bar{\mathcal{E}}\otimes_M\mathcal{E}\to\mathcal{V}$ and $\bar{\mu}_0\colon\mathcal{E}\otimes_M\bar{\mathcal{E}}\to\mathcal{V}$. We have

$$\mu_0(f\psi g)=\varepsilon(f)\mu_0(\psi)\varepsilon(g)\qquad \bar{\mu}_0(f\varphi g)=f\bar{\mu}_0(\varphi)g$$

for each $\psi\in\bar{\mathcal{E}}\otimes_M\mathcal{E}$, $\varphi\in\mathcal{E}\otimes_M\bar{\mathcal{E}}$ and $f,g\in\mathcal{V}$. This implies that there exist invertible smooth functions $U,V\colon M\to\mathbb{C}$ satisfying

$$\mu(\psi)=U\varepsilon^{-1}\mu_0(\psi)\qquad \bar{\mu}(\varphi)=V\bar{\mu}_0(\varphi).$$

Finally, compatibility conditions between μ and $\bar{\mu}$ imply

$$V=\varepsilon(U)\qquad U=U^*.$$

Therefore, quantum line bundles over M are classified by triplets $(L,\varepsilon,U)\leftrightarrow P$ of the above described type.

4.3. Universal Unitary Groups

Let us assume that G is a *universal* [4]-Appendix A compact matrix quantum group, with the fundamental representation u. Let $\mathcal{C}$ be a concrete monoidal W-category [3] generated by elements $\{u,\bar{u}\}$. The objects of $\mathcal{C}$ are just the words over $\{u,\bar{u}\}$, including the unit object. The morphisms between objects of $\mathcal{C}$ are generated by elementary morphisms $\{\gamma^u,\gamma_u,I_u,I^u\}$, and the conjugation maps $\{j_u,j_u^{-1}\}$. The only relations between standard morphisms in $\mathcal{C}$ are given by

$$\begin{gathered}(\gamma^u\otimes\mathrm{id})(\mathrm{id}\otimes I_u)=\mathrm{id}=(\mathrm{id}\otimes\gamma_u)(I^u\otimes\mathrm{id})\\ (\gamma_u\otimes\mathrm{id})(\mathrm{id}\otimes I^u)=\mathrm{id}=(\mathrm{id}\otimes\gamma^u)(I_u\otimes\mathrm{id})\\ \gamma_uI_u=\gamma^uI^u=n_u,\end{gathered}$$

where $n_u=\mathrm{tr}(C_u)=\mathrm{tr}(C_u^{-1})$. We have also the hermicity conditions

$$\begin{aligned}\gamma_u[j_u,j_u^{-1}]&=\gamma_u\qquad &\gamma^u[j_u^{-1},j_u]&=\gamma^u\\ [j_u,j_u^{-1}]I_u&=I_u\qquad &[j_u^{-1},j_u]I^u&=I^u.\end{aligned}$$

Therefore, the representations τ of $\mathcal{C}$ are labeled by conjugate $\mathcal{V}$-bimodules $\mathcal{E}_u,\mathcal{E}_{\bar{u}}$, together with hermitian bimodule injections $\gamma_u^\star$ and $\gamma_\star^u$ and contractions $\langle\rangle_u^{\pm}$, reflecting the above relations in $\mathcal{C}$. In particular, the nature of the structure group influences the whole classification only via the single positive number n_u.

4.4. Some Variations

The presented formalism can be applied to the study of differential structures on quantum principal bundles. Let us assume that the full calculus on P is described by a graded-differential algebra $\Omega(P)$, and let $\mathfrak{hor}(P) \subseteq \Omega(P)$ be the *-subalgebra representing horizontal forms [5]. Let $F^\wedge\colon \mathfrak{hor}(P) \to \mathfrak{hor}(P) \otimes \mathcal{A}$ be the right action map. Let $\Omega(M)$ be a graded-differential *-algebra representing the calculus on M. It consists of $F^\wedge$-invariant elements.

Let $\tau^\star$ be the corresponding representation of $R(G)$ in the category $\Phi_M^\star$ of finite projective $\Omega(M)$-bimodules, with intertwiner bimodules $\mathcal{F}_u$. These bimodules are naturally graded, and $\mathcal{F}_u^0 = \mathcal{E}_u$ for each $u \in \mathcal{E}_u$. Furthermore, the following natural decompositions hold

$$\mathcal{E}_u \otimes_M \Omega(M) \leftrightarrow \mathcal{F}_u \leftrightarrow \Omega(M) \otimes_M \mathcal{E}_u. \tag{4.1}$$

These decompositions are induced by the left/right product maps in $\mathcal{F}_u$.

The composition of the two identifications defines a grade-preserving flip-over operator

$$\sigma_u\colon \mathcal{E}_u \otimes_M \Omega(M) \to \Omega(M) \otimes_M \mathcal{E}_u,$$

acting as the identity on $\mathcal{E}_u$. This operator contains the whole information concerning the $\Omega(M)$-bimodule structure.

The following compatibility condition expresses the consistency of the $\Omega(M)$-bimodule structure. Let m_M be the product in $\Omega(M)$. It follows that

$$\sigma_u(\mathrm{id} \otimes m_M) = (m_M \otimes \mathrm{id})(\mathrm{id} \otimes \sigma_u)(\sigma_u \otimes \mathrm{id}). \tag{4.2}$$

A similar compatibility condition holds between operators σ_u and the natural transformation ϑ. We have

$$(\mathrm{id} \otimes \vartheta_{uv})\sigma_{u\times v} = (\sigma_u \otimes \mathrm{id})(\mathrm{id} \otimes \sigma_v)(\vartheta_{uv} \otimes \mathrm{id}) \tag{4.3}$$

for each $u, v \in R(G)$.

Finally, intertwiner homomorphisms $f_\star$ and conjugation maps $\sharp_u$ satisfy the following diagrams

$$\begin{array}{ccc} \mathcal{E}_v \otimes_M \Omega(M) & \xrightarrow{\sigma_v} & \Omega(M) \otimes_M \mathcal{E}_v \\ {\scriptstyle f_\star \otimes \mathrm{id}}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{id} \otimes f_\star} \\ \mathcal{E}_u \otimes_M \Omega(M) & \xrightarrow[\sigma_u]{} & \Omega(M) \otimes_M \mathcal{E}_u \end{array} \qquad \begin{array}{ccc} \mathcal{E}_u \otimes_M \Omega(M) & \xrightarrow{\sigma_u} & \Omega(M) \otimes_M \mathcal{E}_u \\ {\scriptstyle [\sharp_u, *]}\big\downarrow & & \big\downarrow{\scriptstyle [*, \sharp_u]} \\ \Omega(M) \otimes_M \mathcal{E}_{\bar u} & \xleftarrow[\sigma_{\bar u}]{} & \mathcal{E}_{\bar u} \otimes_M \Omega(M) \end{array}$$

The algebra of horizontal forms can be decomposed in the following way

$$\begin{gathered} \mathfrak{hor}(P) = \sum_{\alpha \in \mathcal{T}}^{\oplus} \mathcal{H}^\alpha(P) \qquad \mathcal{H}^\alpha(P) = \mathcal{F}_\alpha \otimes H_\alpha \\ \mathfrak{hor}(P) \leftrightarrow \Omega(M) \otimes_M \mathcal{B} \leftrightarrow \mathcal{B} \otimes_M \Omega(M). \end{gathered} \tag{4.4}$$

Let us assume that P admits regular connections [5]. For a given regular connection ω, let $D\colon \mathfrak{hor}(P) \to \mathfrak{hor}(P)$ be its covariant derivative. This map is a

right-covariant hermitian first-order antiderivation. In particular, it naturally induces (via compositions) first-order maps $D_u\colon \mathcal{F}_u \to \mathcal{F}_u$. The following properties hold

$$D_{u\times v}(\varphi\otimes\psi) = D_u(\varphi)\psi + (-1)^{\partial\varphi}\varphi D_v(\psi) \qquad D_\varnothing = d_M$$
$$D_{\bar{u}}\sharp_u = \sharp_u D_u \qquad D_u\varrho_{uv}(f) = \varrho_{uv}(f)D_v$$

for each $u,v\in R(G)$ and $f\in \mathrm{Mor}(u,v)$. Here d_M is the differential on $\Omega(M)$.

Conversely, starting from a representation $\tau^\star$ (with graded bimodules, and grade-preserving maps between them) we can reconstruct $\mathfrak{hor}(P)$. If we start from $\mathfrak{hor}(P)$, then the construction of the whole differential calculus on P can be completed applying methods presented in [6]. Covariant derivatives of regular connections are in a natural correspondence with systems $\Bigl\{D_u; u \in R(G)\Bigr\}$ of linear maps $D_u\colon \mathcal{F}_u \to \mathcal{F}_u$ satisfying the above conditions.

References

[1] Connes A: *Non-commutative differential geometry*, Extrait des Publications Mathématiques–IHES **62** (1986)

[2] Woronowicz S L: *Compact matrix pseudogroups*, CMP **111** 613–665 (1987)

[3] Woronowicz S L: *Tannaka-Krein Duality for Compact Matrix Pseudogroups. Twisted $SU(n)$ groups*, Invent Math **93** 35–76 (1988)

[4] Đurđević M: *Geometry of Quantum Principal Bundles I*, Preprint QmmP 6/92, Belgrade University, Serbia (To appear in CMP)

[5] Đurđević M: *Geometry of Quantum Principal Bundles II*, Preprint QmmP 4/93, Belgrade University, Serbia; *Extended version*: Preprint, Instituto de Matematicas, UNAM, México

[6] Đurđević M: *On Differential Structures on Quantum Principal Bundles*, Preprint, Instituto de Matematicas, UNAM, México (1994)

Instituto de Matematicas, UNAM, Area de la Investigacion Cientifica, Circuito Exterior, Ciudad Universitaria, México DF, CP 04510, MEXICO